Wave equation of the scalar field and superfluids

A. Naddeo[*(1)] and G. Scelza[†(2)]

*(1) CNISM, Unità di Ricerca di Salerno and Dipartimento di Fisica "E. R. Caianiello",
Università degli Studi di Salerno, Via Salvador Allende, 84081 Baronissi (SA), Italy
(2) Université de Genève, Departement de Physique Theorique,
24, quai E. Ansermet, CH-1211 Genève, Switzerland*

The new formal analogy between superfluid systems and cosmology, which emerges by taking into account the back-reaction of the vacuum to the quanta of sound waves[1], enables us to put forward some common features between these two different areas of physics. We find the condition that allows us to justify a General Relativity (GR) derivation of the hydrodynamical equation for the superfluid in a four-dimensional space whose metric is the Unruh one[2]. Furthermore we show how, in the particular case taken into account, our hydrodynamical equation can be deduced within a four-dimensional space from the wave equation of a massless scalar field.

I. INTRODUCTION

Recently there has been a growing interest in developing analogue models aimed to probe aspects of the physics of curved space-time and of quantum field theory on curved space-time. Condensed matter analogues have been mainly proposed, because of their conceptual simplicity and experimental feasibility in the laboratory. They appear particularly useful in order to simulate kinematical properties of curved space-time. Such a research area is appealing also because sometimes insights gained within the GR context could help to understand aspects of the analogue model.

Following this line of thought different condensed matter systems have been introduced as analogue models. In such a context acoustics in flowing fluids, light in moving dielectrics or quasiparticles in moving superfluids, have been shown to reproduce some aspects of GR and cosmology[3,4]. They can be conceived as laboratory toy models in order to make experimentally accessible some features of quantum field theory on curved space-time. The analogy between the motion of sound waves in a convergent fluid flow and massless spin-zero particles exposed to a black hole was first outlined in the seminal paper by Unruh[2]. Since then, the search for an emergent space-time has been extended to various media, such as electromagnetic waveguides[5], superfluid helium[3] and Bose-Einstein condensates (BEC)[6]. Emergent space-time and gravity effects in superfluids are of particular interest. Indeed the extremely low temperatures experimentally accessible allow in principle the detection of tiny quantum effects, such as Hawking radiation, particle production and quantum back-reaction[7,8]. In particular, liquid helium II offers the possibility of an experimental control of the value of the speed of first sound, which corresponds to thermal phonons. That may be achieved, for example, by fixing temperature and varying pressure within a wide range (up to about 2.5 MPa) below the lambda transition point[9]. On the other hand, Bose-Einstein condensates made of cold atoms in optical lattices are very promising because of the high degree of experimental control[10]. Indeed such systems have been proposed to mimic an expanding Freedman, Robertson, Walker (FRW) universe[11], where the behavior of quantum modes has been reproduced by manipulating the speed of sound through external fields via Feshbach resonance techniques[12]. The key point of such a finding is the relation $c_s^2 \propto a_s$ between the propagation speed c_s and the s-wave scattering length a_s of the atoms of the condensate[6]. In this way the value of c_s can be changed at will by varying the scattering length in a sufficiently slow manner by means of suitable external fields. That happens without violating the basic assumptions made in deriving the Gross-Pitaevskii equation[10], which describes BEC, and makes some of the predictions of semiclassical quantum gravity and cosmology testable in the laboratory[12].

In a recent paper[1] a new analogy between superfluids and cosmology has been presented, which relies on the depletion of the mass density ρ in the superfluid due to thermal phonons. That is the back-reaction of the vacuum to the quanta of sound waves. Under these new conditions, the free energy is written as the sum of two contributions: the energy of the quantum vacuum and the free energy of the "matter", the phonons. Then, it is possible to show[7] that

$$F(T,\rho) = F_{vac} + F_{mat} = F_{vac} - P_{mat} = \varepsilon(\rho) - \mu\rho - \frac{1}{3}\varepsilon_{mat}(\rho), \qquad (I.1)$$

where $\varepsilon - \mu\rho = \varepsilon_{vac}$ and ε_{mat} are, in this order, the *energy density of the quantum vacuum* and the *energy density*

of the gas of thermal phonons (radiation energy). By expanding the free energy F in terms of $\delta\rho = \rho - \rho_0$ and $\delta\mu = \mu - \mu_0$ (where ρ_0 and μ_0 are the equilibrium density and the chemical potential at $T = 0$, $\rho = \rho(T \neq 0)$ and $\mu = \mu(T \neq 0)$) and then by minimizing over $\delta\rho$ the following hydrodynamical equation[7] is obtained

$$\frac{\delta\rho}{\rho} = -\frac{\varepsilon_{mat}}{\rho c_s^2}\left(\frac{1}{3} + \frac{\rho}{c_s}\frac{\partial c_s}{\partial \rho}\right). \tag{I.2}$$

It is possible to show[1] that, under the hypothesis $\frac{\delta\rho}{\varepsilon_{mat}} << 1$, Eq. (I.2) can be cast in a form which looks very similar to the cosmological fluid one (see Eq. (I.4) for a comparison):

$$\frac{d\rho}{dc_s} \simeq -\frac{3}{c_s}\left(\rho - 3c_s^2\frac{\rho\,\delta\rho}{\varepsilon_{mat}}\right), \tag{I.3}$$

c_s being the phonon speed. Indeed in the cosmological context, the Friedman fluid equation[11]

$$\frac{d\,\rho}{d\,a} = -\frac{3}{a}\left(\rho + \frac{p}{c^2}\right), \tag{I.4}$$

where $a = a(t)$ is the scale parameter of the Universe, is obtained starting from the Einstein equations by means of the condition

$$D_\nu T_0{}^\nu = D_\nu T^{0\nu} = 0, \tag{I.5}$$

where D_ν is the covariant derivative.

Let us notice that the correspondence between the above equations (I.3) and (I.4) is

$$c_s \longleftrightarrow a. \tag{I.6}$$

We point out that the result (I.2), from which we deduce (I.3), can be obtained through the analysis of a classical hydrodynamic equation made by Stone[13] for which the Unruh metric[2] holds. Furthermore the conclusion (I.6) is a crucial one in that leads us to derive in an alternative way the effective metric for the superfluid.

Such a derivation is the aim of this letter. Indeed by exploiting the mathematical analogy between the propagation of sound in a nonhomogeneous potential flow and the propagation of a scalar field in a curved space-time, in full analogy with Ref.[13], we will show that it is possible to introduce an action S and an energy-momentum tensor $T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}$ in such a way that the conservation law $D_\nu T^{\mu\nu} = 0$ is satisfied, D_ν being the usual GR covariant derivative.

In Section 2, following these steps, Eq. (I.3) will be recovered and the formal analogy between GR and fluid dynamics introduced in Ref.[1] will be fully exploited. Finally, in Section 3 we summarize our conclusions and outline some perspectives and open problems.

II. THE FOUR-DIMENSIONAL ANALOGY

In this Section we fully exploit the analogy between the propagation of sound in a nonhomogeneous potential flow and that of a scalar field in a curved space-time by carrying out an analysis similar to the one developed by Stone[13] or Fedichev and Fischer[14]. There it is shown how it is possible to describe a perfect, irrotational fluid, within a four-dimensional formalism, through the equation $\Box\Phi = 0$, where Φ is a scalar field, by means of the Unruh metric.

Let us start by finding an effective metric for the superfluid. The key point of our reasoning is as follows. By looking at the Unruh metric[2]: $ds_{UN}^2 = \frac{\rho(t,\overrightarrow{x})}{c_s(t,\overrightarrow{x})}\left\{-\left[(c_s(t,\overrightarrow{x}))^2 - (\overrightarrow{v}(t,\overrightarrow{x}))^2\right]dt^2 - 2\overrightarrow{v}(t,\overrightarrow{x})\cdot dt\,d\overrightarrow{x} + d\overrightarrow{x}^2\right\}$, where $\overrightarrow{v}(t,\overrightarrow{x})$ is the physical velocity of the superfluid with respect to the laboratory, it is possible to deduce a *minkowskian acoustic metric* in the case $\overrightarrow{v}(t,\overrightarrow{x}) = 0$, i.e. the case of an inner observer. By means of a *conformal transformation*[15], such a metric can be written as

$$[g_{\mu\nu}] = \frac{1}{\rho(t)}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -c_s^{-2}(t) & 0 & 0 \\ 0 & 0 & -c_s^{-2}(t) & 0 \\ 0 & 0 & 0 & -c_s^{-2}(t) \end{pmatrix}. \tag{II.7}$$

Let us notice that the density ρ is a function of the time alone in order to describe an homogeneous fluid.

Now, within the GR context the cosmological fluid equation (I.4) is derived starting from the Einstein equations which do not hold for a quantum fluid. Then, in order to satisfy the condition (I.5) also for the analogue superfluid system

under study we need to proceed in a different way.

Our strategy is the following. Let us suppose that the hydrodynamical equation (I.3) is derived by a wave equation of the kind

$$\Box\Phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi) = 0, \tag{II.8}$$

where $\Phi = \Phi(t,r,\theta,\phi)$ is some massless scalar field and $g_{\mu\nu}$ is the metric tensor (II.7), with $g = Det(g_{\mu\nu})$.

We explicitly note that for a pure radiation field, that is the case considered in (II.8), it is $\frac{p}{c^2} = \frac{\rho}{3}$, then Eq. (I.4) can be written as $\frac{d\rho}{da} = -\frac{4}{a}\rho$. Remembering our hypothesis about $\frac{\delta\rho}{\varepsilon_{mat}}$, we can rewrite (I.3) as $\frac{d\rho}{dc_s} \simeq -\frac{3}{c_s}\rho$ and, then, the formal similitude between Eqs. (I.3) and (I.4) is preserved.

The result is:

$$\Box\Phi = 0 \Rightarrow \frac{d\rho}{dc_s} = -\frac{\rho}{c_s}\left(3 - \frac{c_s}{\dot{c}_s}\frac{\partial_{t,t}\Phi}{\partial_t\Phi} + \frac{c_s^3}{\dot{c}_s\partial_t\Phi}\left(\partial_{\varphi,\varphi}\Phi + \partial_{\theta,\theta}\Phi + \partial_{r,r}\Phi\right)\right) = 0. \tag{II.9}$$

Here, $\dot{c}_s$ is the time derivative of the sound velocity, and ∂_x and $\partial_{x,x}$ are the first and second order partial derivatives with respect to the variable x. Now, by making a simple comparison among Eqs. (II.9) and (I.3), we can also find (with arbitrary parameters) the suitable expression for the field $\Phi(t,r,\theta,\phi)$.

In order to analytically solve such an equation, we make the following simplifying assumptions:

- Φ is a function of the time alone: $\Phi = \Phi(t)$;

- the sound velocity is equal to $c_s(t) = \gamma t^{\frac{1}{2}}$, so that $c_s\dot{c}_s = \frac{\gamma^2}{2}$, γ being a constant[21];

- the quantity $\zeta = \frac{\delta\rho}{\varepsilon_{mat}}$ is assumed to be a constant.

In this way, by means of a direct comparison with Equation (I.3), Equation (II.9) can be easily solved and a simple solution for the field Φ is given by

$$\Phi(t) = \sqrt{2}\,\frac{e^{k\,t}}{k}, \tag{II.10}$$

where $k = \frac{9}{2}\gamma^2\zeta$. Such a field can be identified with the sound field, which is expected to correspond to a quantum coherent state of phonons[13].

The next step to carry out in order to fully exploit the similarity with GR is to find an an action S from which to deduce an energy-momentum tensor that could allow us to achieve Eq. (I.3). Proceeding in analogy with Reference[13], let us introduce an action S defined as:

$$S = \int d^4x \frac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi = \int d^4x\sqrt{-g}\mathcal{L}, \tag{II.11}$$

where the sound field Φ is defined through Eq. (II.8). Let us remember explicitly that such an action gives rise to Eq. (II.8) by setting equal to zero the variation of the action with respect to Φ, i.e. $\delta_\Phi S = 0$ (see for instance Ref.[16]). It is well known[11,17] that any action S automatically provides us with a covariantly conserved and symmetric energy-momentum tensor

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta_g S}{\delta g^{\mu\nu}}, \tag{II.12}$$

where now $\delta_g S$ is the variation of the action with respect to the metric. In this way we find

$$T^{\mu\nu} = \partial^\mu\Phi\partial^\nu\Phi - \frac{1}{2}g^{\mu\nu}(g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi),$$

that, in the case under study, takes the form:

$$[T^{\mu\nu}] = e^{2k\,t}\begin{pmatrix} \rho(t)^2 & 0 & 0 & 0 \\ 0 & c_s(t)^2\rho(t)^2 & 0 & 0 \\ 0 & 0 & c_s(t)^2\rho(t)^2 & 0 \\ 0 & 0 & 0 & c_s(t)^2\rho(t)^2 \end{pmatrix}, \tag{II.13}$$

where the result of Eq. (II.10) has been introduced.

The condition $\delta_g S = 0$, gives rise to the following crucial one[17]:

$$D_\nu T^{\mu\nu} = T^{\mu\nu}{}_{;\nu} = \frac{\partial T^{\mu\nu}}{\partial x^\nu} + \Gamma^\mu_{\nu\sigma}T^{\sigma\nu} + \Gamma^\nu_{\nu\sigma}T^{\mu\sigma} = 0, \tag{II.14}$$

where it appears clearly that D_μ is the usual GR definition of the covariant derivative. In particular, from the equation

$$T^{0\nu}{}_{;\nu} = 0 \tag{II.15}$$

we re-obtain the hydrodynamical equation (I.3). That is a statement of the correctness of our hypotheses and a proof of the formal analogy between GR and superfluid dynamics introduced in Ref.[1].

Furthermore let us notice that, starting from:

$$[T^\mu{}_\nu] = e^{2k\,t} \begin{pmatrix} \rho(t) & 0 & 0 & 0 \\ 0 & -\rho(t) & 0 & 0 \\ 0 & 0 & -\rho(t) & 0 \\ 0 & 0 & 0 & -\rho(t) \end{pmatrix}, \tag{II.16}$$

the energy density and the pressure of the massless scalar field Φ are derived and take the form (we follow the definition of Kolb and Turner[18]):

$$\rho_\Phi = T^0{}_0 = e^{2k\,t}\rho(t)$$

and

$$p_\Phi = \frac{1}{3}T^i{}_i = -e^{2k\,t}\rho(t).$$

Then, finally, we automatically get the relation

$$\rho_\Phi = -p_\Phi, \tag{II.17}$$

which coincides with the equation of state for the inflaton field in an inflationary universe[19]. We plan to further investigate this topic and to clarify how inflationary dynamics can be mimicked in the laboratory in a future publication[20].

III. CONCLUSIONS AND PERSPECTIVES

In conclusion, by means of the field equation (II.9) we have rewritten the hydrodynamical equation (I.3) within a four-dimensional framework. That allowed us to make a direct comparison among superfluid dynamics and GR. Following Stone[13] we were able to introduce an action S and an energy-momentum tensor $T^{\mu\nu}$ for the sound field Φ. That allowed us to derive the relevant hydrodynamical equation (I.3) from the condition $D_\nu T^{\mu\nu} = 0$ even if superfluid dynamics theory lacks general covariance. In this way the analogy between GR and superfluid theory can be made very transparent. We stress that within the new conditions considered here, i. e. the back-reaction of the vacuum to the quanta of sound waves, we are faced with a sound speed c_s depending on the time. In order to make Equation $\Box\Phi = 0$ equivalent to Equation (I.3), the relation $c_s \propto t^{\frac{1}{2}}$ has been found.

The relevance of finding similarities among different fields of research and of building up analogue models in order to test theories which otherwise could not be proven. In the particular case we study in this letter, the paradigm of cosmology as a research area where it is not possible to check hypotheses is bypassed since we have a toy model, a superfluid system built in the laboratory, where our predictions could be tested. Then, all the formal similarities we find could give us new insights to understand the nature of the universe. For instance, in the context of superfluid systems the presence of thermal phonons plays the same role as the matter in the universe. At this stage a question seems to emerge as a speculation: within the cosmological area, can we conceive the matter as a thermal perturbation of the vacuum as for the superfluids?

IV. ACKNOWLEDGEMENTS

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[21] let us remember that the cosmological analogue $a(t) \propto t^{\frac{1}{2}}$ holds for an Universe *radiation dominated*.